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Quasi-exact solvability of Inozemtsev models

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Abstract

Finite-dimensional spaces which are invariant under the action of the Hamiltonian of the BC_N Inozemtsev model are introduced, and it is shown that the commuting operators of conserved quantities also preserve the finite-dimensional spaces. The relationship between the finite-dimensional spaces of the BC_N Inozemtsev models and the theta-type invariant spaces of the BC_N Ruijsenaars models is clarified. The degeneration of the BC_N Inozemtsev models and the correspondence of their invariant spaces are considered.

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1. Introduction

In [6], Inozemtsev proposed an N-particle quantum mechanics model, which is called the BC_N Inozemtsev model. This is a generalization of the Calogero–Moser–Sutherland model or the Olshanetsky–Perelomov model [12].

The BC_N Inozemtsev model is quantum completely integrable. Here, quantum complete integrability means that there exist N algebraically independent mutually commuting operators (higher commuting Hamiltonians) which commute with the Hamiltonian of the model. It is a quantum version of Liouville's integrability. For the BC_N Inozemtsev model, Oshima [13] described the commuting operators explicitly. Note that the BC_N Inozemtsev model is a universal completely integrable model of quantum mechanics with B_N symmetry, which follows from the classification due to Ochiai, Oshima and Sekiguchi [11, 14].

On the other hand, Finkel, Gomez-Ullate, Gonzalez-Lopez, Rodriguez and Zhdanov studied quasi-exactly solvable models in [2, 3]. They found several quasi-exactly solvable many-body systems. Although they did not use the phrase 'BC_N Inozemtsev model', they essentially found that the BC_N Inozemtsev model is quasi-exactly solvable, i.e. the Hamiltonian of the BC_N Inozemtsev model preserves some finite-dimensional space which is spanned by some symmetric 'monomials'.

In this paper, we link the quasi-exact solvability with the quantum complete integrability. More precisely, we show that the commuting operators (higher Hamiltonians) of the BC_N Inozemtsev model also preserve the finite-dimensional space, which has appeared in the context of quasi-exact solvability.

On the finite-dimensional space, joint eigenvalues and eigenfunctions of the commuting operators are determined by algebraic calculations. In this sense, the model would be solved partially. Note that the phrase 'quasi-exact solvability' is used in these situations (see [21]).

The spectral problem of quantum mechanics is generally considered in a Hilbert space, and the Hilbert space is often taken as a square-integrable space (L^2 space). Therefore, it would be important to consider the relationship between the Hilbert space (L^2 space) of the BC_N Inozemtsev model and the finite-dimensional space which appears in the context of the quasi-exact solvability. In this paper, we determine the condition that the finite-dimensional space lies in the L^2 space.

There are other models which are concerned with the quasi-exact solvability. Ruijsenaars-type models are introduced for arbitrary root systems including the BC_N cases, which are difference analogues of Inozemtsev (or Calogero–Moser–Sutherland) models (see [1, 5, 15]). Hikami and Komori [5, 7–9] finally constructed higher commuting operators using root algebra, and found an invariant subspace spanned by theta functions. In [16], Sasaki and Takasaki considered degenerate Inozemtsev models and their quasi-exact solvability.

In this paper, correspondence between the BC_N Ruijsenaars model and the BC_N Inozemtsev model is considered. We will observe that the invariant subspace spanned by theta functions for the BC_N Ruijsenaars model corresponds to the invariant space related to the quasi-exact solvability for the BC_N Inozemtsev model.

To obtain the degenerate BC_N Inozemtsev model from the (elliptic) BC_N Inozemtsev model, a certain trigonometric limit is considered. It is shown that the finite-dimensional invariant spaces for the BC_N Inozemtsev model tend to the invariant spaces of the degenerate BC_N Inozemtsev model which were introduced by Sasaki and Takasaki. It would be important to consider the degeneration of models, because it would be helpful in understanding several integrable models and the relationship among them.

This paper is organized as follows. In section 2, the BC_N Inozemtsev model and its finite-dimensional invariant spaces are introduced. In section 3.1, higher commuting operators of the BC_N Inozemtsev model are introduced. In section 3.2, it is shown that the higher commuting operators also preserve the finite-dimensional invariant spaces. In section 3.3, the relationship between the finite-dimensional invariant spaces and the L^2 space is considered. If the coupling constants are integers, the model may have some special features. In section 3.4, we consider this case. In section 4, the correspondence between the theta-type invariant spaces for the BC_N Ruijsenaars model and the invariant spaces which is related to the quasi-exact solvability for the BC_N Inozemtsev model is investigated. In section 5, we consider the degeneration of the BC_N Inozemtsev models and the limit of invariant spaces.

2. BC_N Inozemtsev model and its invariant space

The BC_N Inozemtsev model is a system of quantum mechanics with N-particles whose Hamiltonian is given by

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2l(l+1) \sum_{1\leqslant j<k\leqslant N} (\wp(x_j - x_k) + \wp(x_j + x_k)) + \sum_{j=1}^{N}\sum_{i=0}^{3} l_i(l_i+1)\wp(x_j + \omega_i) \tag{2.1}$$

where $\wp(x)$ is the Weierstrass' $\wp$-function with periods $(1, \tau)$ (see (A.1)), $\omega_0 = 0, \omega_1 = \frac{1}{2}$, $\omega_2 = -\frac{\tau+1}{2}, \omega_3 = \frac{\tau}{2}$ are half-periods, and l and l_i $(i = 0, 1, 2, 3)$ are coupling constants.

Let a, b_i $(i = 0, 1, 2, 3)$ be numbers which satisfy $a \in \{-l, l+1\}$ and $b_i \in \left\{-\frac{l_i}{2}, \frac{l_i+1}{2}\right\}$ $(i = 0, 1, 2, 3)$. Set $z_j = \wp(x_j)$ $(1 \leqslant j \leqslant N)$ and

$$\Phi(z) = \prod_{1\leqslant j<k\leqslant N} (z_j - z_k)^a \prod_{j=1}^{N}\prod_{i=1}^{3} (z_j - e_i)^{b_i} \qquad \widehat{H} = \Phi(z)^{-1} \circ H \circ \Phi(z) \tag{2.2}$$

where $e_i = \wp(\omega_i)$ $(i = 1, 2, 3)$.

By applying formulae (A.3), it is shown directly that the operator $\widehat{H}$ admits the following expression:

$$\begin{aligned}\widehat{H} = &-\left(\sum_{j=1}^{N} 4(z_j - e_1)(z_j - e_2)(z_j - e_3)\left(\frac{\partial^2}{\partial z_j^2} + \left(\sum_{k\neq j} \frac{2a}{z_j - z_k} + \frac{2b_1 + \frac{1}{2}}{z_j - e_1} + \frac{2b_2 + \frac{1}{2}}{z_j - e_2}\right.\right.\right.\\ &\left.\left.\left. + \frac{2b_3 + \frac{1}{2}}{z_j - e_3}\right)\frac{\partial}{\partial z_j}\right)\right) - 4\left((N-1)a - b_0 + b_1 + b_2 + b_3 + \frac{1}{2}\right)((N-1)a + b_0\\ &+ b_1 + b_2 + b_3)\left(\sum_{j=1}^{N} z_j\right) + 4N((b_1 + b_2)^2 e_3 + (b_1 + b_3)^2 e_2 + (b_2 + b_3)^2 e_1)\\ &- 4N(N-1)a(e_1 b_1 + e_2 b_2 + e_3 b_3).\end{aligned} \tag{2.3}$$

Proposition 2.1.

(i) Let P^{sym} be the space of symmetric polynomials in variables $z_1, z_2, \ldots, z_N$. Then $\widehat{H} \cdot P^{\mathrm{sym}} \subset P^{\mathrm{sym}}$.

(ii) Let a, b_i $(i = 0, 1, 2, 3)$ be numbers which satisfy $a \in \{-l, l+1\}$ and $b_i \in \left\{-\frac{l_i}{2}, \frac{l_i+1}{2}\right\}$ $(i = 0, 1, 2, 3)$. Assume that $d = -((N-1)a + b_0 + b_1 + b_2 + b_3)$ is a non-negative integer. Let V_d be the vector space spanned by monomials $z_1^{m_1} z_2^{m_2} \ldots z_N^{m_N}$ such that $m_i \in \{0, 1, \ldots, d\}$ for all i, and $V_d^{\mathrm{sym}} = V_d \cap P^{\mathrm{sym}}$. Then $\widehat{H} \cdot V_d^{\mathrm{sym}} \subset V_d^{\mathrm{sym}}$.

Proof. Let $f \in P^{\mathrm{sym}}$. From (2.3), the function $\widehat{H} f$ is a symmetric rational function and it may have poles only along $z_j - z_k = 0$ of degree at most 1. If $\widehat{H} f$ has a pole along $z_j - z_k = 0$ $(j \neq k)$ of degree 1, it contradicts the symmetry of $\widehat{H} f$ on the variables z_j and z_k. Hence, the function $\widehat{H} f$ does not have poles and we obtain (i).

Let $\lambda_1, \ldots, \lambda_N$ be non-negative integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N \geqslant 0$.

Set $P(z) = 4(z - e_1)(z - e_2)(z - e_3)$ and $L = \lambda_1$. Let l and l' be integers such that $0 \leqslant l, l' \leqslant L$. Then

$$\begin{aligned}&\left(\frac{P(z_j)}{(z_j - z_k)}\frac{\partial}{\partial z_j} + \frac{P(z_k)}{(z_k - z_j)}\frac{\partial}{\partial z_k}\right)\left(z_j^l z_k^{l'} + z_j^{l'} z_k^l\right)\\ &\qquad = 4L\delta_{l,L}\left(z_j^{L+1} z_k^{l'} + z_j^{l'} z_k^{L+1}\right) + 4L\delta_{l',L}\left(z_j^l z_k^{L+1} + z_j^{L+1} z_k^l\right) + (\#_1)\end{aligned} \tag{2.4}$$

where $\delta_{l,L}$ is Kronecker's delta and the term $(\#_1)$ is a linear combination of monomials $z_j^t z_k^{t'}$ such that $0 \leqslant t, t' \leqslant L$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ and $\lambda^+ = (\lambda_1 + 1, \lambda_2, \dots, \lambda_N)$. By summing up the equality (2.4) on j and k $(1 \leqslant j < k \leqslant N)$, we obtain

$$\sum_{j=1}^{N} \sum_{k \neq j} \frac{P(z_j)}{(z_j - z_k)} \frac{\partial}{\partial z_j} m_\lambda = 4(N-1)Lm_{\lambda^+} + (\#_2) \tag{2.5}$$

where $m_\mu = \sum_{(m_1,\dots,m_N) \in \mathfrak{S}_N \cdot \mu} z_1^{m_1} z_2^{m_2} \dots z_N^{m_N}$ for $\mu = (\mu_1, \dots, \mu_N)$, $\mathfrak{S}_N$ is the symmetric group of N letters and the term $(\#_2)$ is a linear combination of symmetric monomials $m_{(\mu_1,\dots,\mu_N)}$ such that $0 \leqslant \mu_N \leqslant \dots \leqslant \mu_1 \leqslant L$.

Hence we obtain

$$\widehat{H} m_\lambda = -4\left(L - d - 2b_0 + \tfrac{1}{2}\right)(L-d) m_{\lambda^+} + (\#_3) \tag{2.6}$$

where the term $(\#_3)$ is a linear combination of symmetric monomials $m_{(\mu_1,\dots,\mu_N)}$ such that $0 \leqslant \mu_N \leqslant \dots \leqslant \mu_1 \leqslant L$.

If $L \leqslant d$ then all elements in the term $(\#_3)$ lie in V_d^{sym}. If $L < d$ then $m_{\lambda^+} \in V_d^{\mathrm{sym}}$, and if $L = d$ then the coefficient of m_{λ^+} on the right-hand side of (2.6) vanishes. Hence if $L \leqslant d$ then $\widehat{H} m_\lambda \in V_d^{\mathrm{sym}}$.

Therefore, we obtain (ii). □

Remark. Proposition 2.1 was essentially obtained in [3] by a different method.

3. Commuting operators and invariant subspaces

3.1. Commuting operators

It is known that the BC_N Inozemtsev model is completely integrable, i.e. there exist N algebraically independent mutually commuting operators (higher commuting Hamiltonians) which commute with the Hamiltonian. In [13], Oshima gave explicit forms of the commuting operators. Now we pick up some results obtained by Oshima.

Let $W(B_N)$ be the Weyl group of type B_N, i.e. the group of coordinate transformations

$$(x_1, \dots, x_N) \mapsto (\epsilon_1 x_{\sigma(1)}, \dots, \epsilon_N x_{\sigma(N)}) \tag{3.1}$$

of $\mathbb{R}^N$, where $\sigma \in \mathfrak{S}_N$ (the symmetric group) and $\epsilon_1 = \pm 1, \dots, \epsilon_N = \pm 1$. Let $W(D_N)$ be a subgroup of $W(B_N)$ which consists of transformations (3.1) with a condition $\prod_{j=1}^N \epsilon_j = 1$. For $w \in W(B_N)$ we define $\epsilon(w) = \begin{cases} 1 & w \in W(D_N) \\ -1 & w \notin W(D_N) \end{cases}$.

Let us consider the operators which commute with the Hamiltonian of the BC_N Inozemtsev model (2.1). For $i = 0, 1, 2, 3$, we set

$$\begin{aligned}
S_{\{1,\dots,k\}} &= \sum_{w \in W(B_k)} w\left(\wp(x_1 - x_2)\wp(x_2 - x_3)\cdots\wp(x_{k-1} - x_k)\right) \\
S^{\langle i \rangle}_{\{1,\dots,k\}} &= \sum_{w \in W(B_k)} w\left(\wp(x_1 + \omega_i)\wp(x_1 - x_2)\wp(x_2 - x_3)\cdots\wp(x_{k-1} - x_k)\right) \\
T^{o}_{\{1,\dots,k\}} &= \sum_{I_1 \amalg \dots \amalg I_\mu = \{1,\dots,k\}} (-1)^{\mu-1}(\mu-1)!\, S_{I_1} \dots S_{I_\mu} \\
T^{o,\langle i \rangle}_{\{1,\dots,k\}} &= \sum_{I_1 \amalg \dots \amalg I_\mu = \{1,\dots,k\}} (-1)^{\mu-1}(\mu-1)!\, S^{\langle i \rangle}_{I_1} \dots S^{\langle i \rangle}_{I_\mu} \\
T_{\{1,\dots,k\}} &= -(-l(l+1))^{k-1} \sum_{i=0}^{3} \frac{l_i(l_i+1)}{2} T^{o,\langle i \rangle}_{\{1,\dots,k\}}
\end{aligned} \tag{3.2}$$

where the sum $\sum_{I_1 \amalg \cdots \amalg I_\mu = \{1,\ldots,k\}}$ runs over all different partitions of $\{1, \ldots, k\}$. For example

$$T^o_\phi = 1 \qquad T^o_{\{1\}} = S_{\{1\}} \qquad T^o_{\{1,2\}} = S_{\{1,2\}} - S_{\{1\}} S_{\{2\}}$$
$$T^o_{\{1,2,3\}} = S_{\{1,2,3\}} - S_{\{1\}} S_{\{2,3\}} - S_{\{2\}} S_{\{1,3\}} - S_{\{3\}} S_{\{1,2\}} + 2 S_{\{1\}} S_{\{2\}} S_{\{3\}}.$$

Set

$$\begin{aligned} \Delta_{\{1,\ldots,k\}} &= \sum_{0 \leqslant j \leqslant [\frac{k}{2}]} \frac{(l(l+1))^j}{2^k j!(k-2j)!} \sum_{w \in W(B_k)} \epsilon(w) w \Bigg(\wp(x_1 - x_2) \wp(x_3 - x_4) \cdots \\ &\quad \times \wp(x_{2j-1} - x_{2j}) \frac{\partial}{\partial x_{2j+1}} \frac{\partial}{\partial x_{2j+2}} \cdots \frac{\partial}{\partial x_k} \Bigg) \\ q_{\{1,\ldots,k\}} &= \sum_{I_1 \amalg \cdots \amalg I_\mu \in \{1,\ldots,k\}} T_{I_1} \ldots T_{I_\mu} \\ T_{w(\{1,\ldots,k\})} &= w(T_{\{1,\ldots,k\}}) \qquad \Delta_{w(\{1,\ldots,k\})} = w(\Delta_{\{1,\ldots,k\}}) \qquad \text{for} \quad w \in \mathfrak{S}_N \end{aligned} \tag{3.3}$$

where $\left[\frac{k}{2}\right]$ represents the maximum integer not greater than $\frac{k}{2}$.

Proposition 3.1 ([13] theorem 7.2, remark 7.4)**.** *Set*

$$\begin{aligned} P_{N-k} &= \sum_{i=k}^{N} \sum_{j=i}^{N} \frac{1}{i!(j-i)!(N-j)!} \sum_{w \in \mathfrak{S}_N} \sum_{I_1 \amalg \cdots \amalg I_k = \{1,\ldots,i\}} \\ &\quad \times w \left((-l(l+1))^{i-k} 2^{-k} T^o_{I_1} \ldots T^o_{I_k} q_{\{i+1,\ldots,j\}} \Delta^2_{\{j+1,\ldots,N\}} \right) \end{aligned} \tag{3.4}$$

for $k = 0, \ldots, N-1$. Then the operators $P_j (1 \leqslant j \leqslant N)$ are $W(B_N)$-invariant and

$$[P_j, P_k] = 0 \tag{3.5}$$

for $1 \leqslant j, k \leqslant N$. The Hamiltonian H (2.1) is a linear combination of P_1 and 1, i.e. $H = AP_1 + B$ for some constants A, B.

3.2. Commuting operators and invariant subspaces

We change variables by $z_j = \wp(x_j)$ $(j = 1, \ldots, N)$ and set

$$\widehat{P}_k = \Phi(z)^{-1} \circ P_k \circ \Phi(z) \qquad (k = 1, \ldots, N) \tag{3.6}$$

where $\Phi(z)$ is defined in (2.2). From (3.5), we obtain

$$[\widehat{P}_j, \widehat{P}_k] = 0 \tag{3.7}$$

for $1 \leqslant j, k \leqslant N$.

Proposition 3.2. *The operators $\widehat{P}_k$ $(k = 1, \ldots, N)$ admit the following expansion:*

$$\widehat{P}_k = \sum_{\substack{0 \leqslant i_1,\ldots,i_N \leqslant 2 \\ i_1 + \cdots + i_N \leqslant 2k}} c_{i_1,\ldots,i_N}(z) \left(\frac{\partial}{\partial z_1} \right)^{i_1} \cdots \left(\frac{\partial}{\partial z_N} \right)^{i_N}. \tag{3.8}$$

Here, the operators $\widehat{P}_k$ are symmetric in $z_1, \ldots, z_N$ and the coefficients $c_{i_1,\ldots,i_N}(z)$ are rational functions in $z_1, \ldots, z_N$ which may have poles only along $z_j - e_i = 0$ and $z_{j_1} - z_{j_2} = 0$ $(1 \leqslant i \leqslant 3, 1 \leqslant j, j_1, j_2 \leqslant N, j_1 \neq j_2)$.

Proof. From (3.4), the operator P_k admits the expansion

$$P_k = \sum_{\substack{0\leqslant i_1,\dots,i_N\leqslant 2\\ i_1+\cdots+i_N\leqslant 2k}} d_{i_1,\dots,i_N}(x_1,\dots,x_N)\left(\frac{\partial}{\partial x_1}\right)^{i_1}\cdots\left(\frac{\partial}{\partial x_N}\right)^{i_N} \tag{3.9}$$

such that $d_{i_1,\dots,i_N}(x_1,\dots,x_N)$ is doubly periodic for each variable $x_1,\dots,x_N$. From the $W(B_N)$-symmetry of the operator P_k, we obtain $d_{i_1,\dots,i_N}(x_1,\dots,-x_j,\dots,x_N) = (-1)^{i_j} d_{i_1,\dots,i_N}(x_1,\dots,x_j,\dots,x_N)$ for $j = 1,\dots,N$.

Since $\wp'(x)$ is an odd doubly periodic function, the function $d^e_{i_1,\dots,i_N}(x_1,\dots,x_N)$ defined by $d_{i_1,\dots,i_N}(x_1,\dots,x_N) = d^e_{i_1,\dots,i_N}(x_1,\dots,x_N)\prod_{j;\,i_j=1}\wp'(x_j)$ is even doubly periodic in each x_j $(j = 1,\dots,N)$.

Now, change variables $z_j = \wp(x_j)$ $(j = 1,\dots,N)$ and write

$$P_k = \sum_{\substack{0\leqslant i_1,\dots,i_N\leqslant 2\\ i_1+\cdots+i_N\leqslant 2k}} \tilde{d}_{i_1,\dots,i_N}(x_1,\dots,x_N)\left(\frac{\partial}{\partial z_1}\right)^{i_1}\cdots\left(\frac{\partial}{\partial z_N}\right)^{i_N}. \tag{3.10}$$

Since $\wp'(x)^2$ and $\wp''(x)$ are even doubly periodic, the function $\tilde{d}_{i_1,\dots,i_N}(x_1,\dots,x_N)$ is even doubly periodic in each x_j $(j = 1,\dots,N)$. It is known that an even doubly periodic function in x is expressed as $F(\wp(x))$ with some rational function $F(z)$. Hence, if we set $z_j = \wp(x_j)$ $(j = 1,\dots,N)$, the functions $\tilde{d}_{i_1,\dots,i_N}(x_1,\dots,x_N)$ admit the expression $\tilde{d}_{i_1,\dots,i_N}(x_1,\dots,x_N) = \tilde{c}_{i_1,\dots,i_N}(\wp(x_1),\dots,\wp(x_N))$ with some rational function $\tilde{c}_{i_1,\dots,i_N}(z_1,\dots,z_N)$. From equality (3.4) and formulae (A.3), it is shown that the coefficients $\tilde{c}_{i_1,\dots,i_N}(z_1,\dots,z_N)$ may have poles only along $z_j - e_i = 0$ and $z_{j_1} - z_{j_2} = 0$ $(1 \leqslant i \leqslant 3, 1 \leqslant j, j_1, j_2 \leqslant N, j_1 \neq j_2)$.

Next we consider the coefficients of the operators $\widehat{P}_k$ defined by (3.6). From properties of the functions $\tilde{c}_{i_1,\dots,i_N}(z_1,\dots,z_N)$, it is shown that the coefficients $c_{i_1,\dots,i_N}(z)$ in (3.8) are rational functions in variables $z_1,\dots,z_N$ which may have poles only along $z_j - e_i = 0$ and $z_{j_1} - z_{j_2} = 0$ $(1 \leqslant i \leqslant 3, 1 \leqslant j, j_1, j_2 \leqslant N, j_1 \neq j_2)$.

Since the operators P_k and the function $\Phi(z)$ are symmetric in $z_1,\dots,z_N$, the operators $\widehat{P}_k$ are symmetric in $z_1,\dots,z_N$. □

Theorem 3.3.

(i) Let P^{sym} be the space of symmetric polynomials in variables $z_1,\dots,z_N$, then $\widehat{P}_k \cdot P^{\mathrm{sym}} \subset P^{\mathrm{sym}}$ for $k = 1, 2, \dots, N$.

(ii) Let a, b_i $(i = 0, 1, 2, 3)$ be numbers which satisfy $a \in \{-l, l+1\}$ and $b_i \in \left\{-\frac{l_i}{2}, \frac{l_i+1}{2}\right\}$ $(i = 0, 1, 2, 3)$. Assume that $d = -((N-1)a + b_0 + b_1 + b_2 + b_3)$ is a non-negative integer. Then $\widehat{P}_k \cdot V^{\mathrm{sym}}_d \subset V^{\mathrm{sym}}_d$ for $k = 1, 2, \dots, N$, where V^{sym}_d is the finite-dimensional space defined in proposition 2.1.

Proof. The coefficients $c_{i_1,\dots,i_N}(z)$ in (3.8) are rational functions which may have poles only along $z_j - e_i = 0, z_{j_1} - z_{j_2} = 0$ $(1 \leqslant i \leqslant 3, 1 \leqslant j, j_1, j_2 \leqslant N, j_1 \neq j_2)$.

Let us fix j_1 and j_2 that satisfy $j_1 \neq j_2$. Let p be the maximal number of degrees of a pole along $z_{j_1} - z_{j_2} = 0$ of functions $c_{i_1,\dots,i_N}(z)$ for all possible $i_1,\dots,i_N$. If $f(z) \in P^{\mathrm{sym}}$ then the function $\widehat{P}_k f(z)$ has a pole at most degree p along $z_{j_1} - z_{j_2} = 0$. Let $f(z)$ be an element of P^{sym} such that the function $\widehat{P}_k f(z)$ has a pole of maximum degree along $z_{j_1} - z_{j_2} = 0$. We denote the degree by p'. Then the function $\widehat{P}_k \widehat{H} f(z)$ has a pole of degree at most p' along $z_{j_1} - z_{j_2} = 0$, because $\widehat{H} f(z) \in P^{\mathrm{sym}}$. On the other hand, it is shown that the function $\widehat{H}\widehat{P}_k f(z)$ has a pole of degree $p'+2$ along $z_{j_1} - z_{j_2} = 0$ if $p' \neq 0, 1-2a$. Since $\widehat{P}_k \widehat{H} f(z) = \widehat{H}\widehat{P}_k f(z)$, if $1-2a \notin \mathbb{Z}$ then p' must be equal to zero. Hence the function $\widehat{P}_k f(z)$ is holomorphic along $z_{j_1} - z_{j_2} = 0$ if $1 - 2a \notin \mathbb{Z}$. By a continuity argument in a, it is shown that the function $\widehat{P}_k f(z)$ is holomorphic along $z_{j_1} - z_{j_2} = 0$ for all a.

Similarly if $f(z) \in P^{\mathrm{sym}}$ then the function $\widehat{P}_k f(z)$ is holomorphic along $z_j - e_i = 0$ and $z_{j_1} - z_{j_2} = 0$ $(1 \leqslant i \leqslant 3, 1 \leqslant j, j_1, j_2 \leqslant N, j_1 \neq j_2)$. Hence $\widehat{P}_k f(z) \in P^{\mathrm{sym}}$.

Next we prove (ii).

From expression (3.8), there exists $p \in \mathbb{Z}_{\geqslant 0}$ such that $\widehat{P}_k f(z) \subset V^{\mathrm{sym}}_{d+p}$ for all $f(z) \in V^{\mathrm{sym}}_d$. Let $f(z)$ be an element of V^{sym}_d such that the degree of $\widehat{P}_k f(z)$ is maximum. We denote the degree by $d + p'$. Then the function $\widehat{P}_k \widehat{H} f(z)$ has a degree at most $d + p'$, because $\widehat{H} f(z) \in V^{\mathrm{sym}}_d$. On the other hand, it is shown that the function $\widehat{H}\widehat{P}_k f(z)$ has a degree $d + p' + 1$ if $p' \neq 0,\ 2b_0 - 1/2$. From the commutativity of $\widehat{H}$ and $\widehat{P}_k$, if $2b_0 - 1/2 \notin \mathbb{Z}$ then p' must be equal to zero. Thus $\widehat{P}_k f(z) \in V^{\mathrm{sym}}_d$. By a continuity argument, we can remove the condition $2b_0 - 1/2 \notin \mathbb{Z}$.

Hence we obtain (ii). □

In summary, we established that the higher commuting Hamiltonians also preserve the space related to the quasi-exact solvability in theorem 3.3 (ii).

3.3. Relationship to the L^2 space

Assume $l, l_0, l_1 \in \mathbb{R}_{\geqslant 0}$ and $l_2, l_3 \in \mathbb{R}$ in this subsection.

The invariant space V^{sym}_d of the BC_N Inozemtsev model is defined for each $a \in \{-l, l+1\}$ and $b_i \in \left\{-\frac{l_i}{2}, \frac{l_i+1}{2}\right\}(i = 0, 1, 2, 3)$ with the condition $d = -((N-1)a + b_0 + b_1 + b_2 + b_3) \in \mathbb{Z}_{\geqslant 0}$. For these numbers a, b_0, b_1, b_2, b_3, define

$$W^{\mathrm{sym}}_d = \left\{\Phi(\wp(x_1), \ldots, \wp(x_N)) f(\wp(x_1), \ldots, \wp(x_N)) | f(z_1, \ldots, z_N) \in V^{\mathrm{sym}}_d\right\} \tag{3.11}$$

where the function $\Phi(z_1, \ldots, z_N) = \prod_{1 \leqslant j<k \leqslant N} (z_j - z_k)^a \prod_{j=1}^{N} \prod_{i=1}^{3} (z_j - e_i)^{b_i}$ was defined in (2.2).

From relations (2.2), (3.6) and theorem 3.3, the following proposition is shown immediately:

Proposition 3.4. *Assume that $d = -((N-1)a + b_0 + b_1 + b_2 + b_3)$ is a non-negative integer. Then $H \cdot W^{\mathrm{sym}}_d \subset W^{\mathrm{sym}}_d$ and $P_k \cdot W^{\mathrm{sym}}_d \subset W^{\mathrm{sym}}_d$ for $k = 1, 2, \ldots, N$.*

We look into the condition that the space W^{sym}_d lies in L^2 space. If $f(x) \in W^{\mathrm{sym}}_d$, then $|f(x)| \sim |x_j - x_k|^a$ (resp. $|f(x)| \sim |x_j + x_k|^a$) as $|x_j - x_k| \to 0$ (resp. $|x_j + x_k| \to 0$) $(j \neq k)$, $|f(x)| \sim |x_j|^{2b_0}$ as $|x_j| \to 0$, and $|f(x)| \sim \left|x_j - \frac{1}{2}\right|^{2b_1}$ as $\left|x_j - \frac{1}{2}\right| \to 0$. Hence if $a = l+1$, $b_0 = \frac{l_0+1}{2}$ and $b_1 = \frac{l_1+1}{2}$ then the function $f(x) \in W^{\mathrm{sym}}_d$ is locally square-integrable on $|x_j - x_k|, |x_j + x_k|, |x_j|, \left|x_j - \frac{1}{2}\right| < \epsilon$ for sufficiently small ϵ. Combining with the periodicity of $f(x) \in W^{\mathrm{sym}}_d$, we obtain $\int_{0<x_1<\cdots<x_N<1} |f(x)|^2 \,\mathrm{d}x_1 \cdots \mathrm{d}x_N < \infty$. Hence the following proposition is shown.

Proposition 3.5. *Let $b_i \in \left\{-\frac{l_i}{2}, \frac{l_i+1}{2}\right\}$ $(i = 2, 3)$. If $d = -\left((N-1)(l+1) + \frac{l_0+l_1}{2} + 1 + b_2 + b_3\right) \in \mathbb{Z}_{\geqslant 0}$ then every function in W^{sym}_d is square-integrable on the domain $0 < x_1 < \cdots < x_N < 1$.*

In the case $d = -\left((N-1)(l+1) + \frac{l_0+l_1}{2} + 1 + b_2 + b_3\right) \in \mathbb{Z}_{\geqslant 0}$, some eigenvalues of the commuting Hamiltonians on the Hilbert space (L^2 space) appear as the eigenvalues on the subspace W^{sym}_d. Hence some eigenvalues on the Hilbert space would be obtained explicitly, because the eigenvalues in the finite-dimensional space are obtained by algebraic calculations. Note that the case BC_1 was done in [18], and Gomez-Ullate, Gonzalez-Lopez and Rodriguez considered the relationship between the L^2 space and the space related to the quasi-exact solvability for some special cases in [4].

As an aside, the joint eigenvalues of the trigonometric BC_N Calogero–Sutherland model are already known and their expression is simple. Distributions of eigenvalues in $L^2 \cap W_d^{\mathrm{sym}}$ will be detected by considering the trigonometric limit $\tau \to \sqrt{-1}\infty$ while fixing coupling constants l, l_0, l_1, l_2, l_3.

3.4. The case $l, l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geqslant 0}$

Let us consider the case $l, l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geqslant 0}$. In this case, the Hamiltonian and the higher commuting Hamiltonians preserve several finite-dimensional spaces of elliptic functions.

The invariant space W_d^{sym} is defined for each $a \in \{-l, l+1\}$ and $b_i \in \left\{-\frac{l_i}{2}, \frac{l_i+1}{2}\right\}$ $(i = 0, 1, 2, 3)$ with the condition $d = -((N-1)a + b_0 + b_1 + b_2 + b_3) \in \mathbb{Z}_{\geqslant 0}$.

For each $l, l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geqslant 0}$, there are eight possible sets of (a, b_0, b_1, b_2, b_3) for which the invariant space W_d^{sym} is defined, if $N \geqslant 2$. If $N = 1$ then there are four possible sets of (b_0, b_1, b_2, b_3). For example, if $N \geqslant 2$, $l \gg l_0, l_1, l_2, l_3$ and $(N-1)l + l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}_{>0}$, then the cases

$$(a, b_0, b_1, b_2, b_3) = \begin{cases} \left(-l, \frac{-l_0}{2}, \frac{-l_1}{2}, \frac{-l_2}{2}, \frac{-l_3}{2}\right), \left(-l, \frac{l_0+1}{2}, \frac{l_1+1}{2}, \frac{l_2+1}{2}, \frac{l_3+1}{2}\right), \\ \left(-l, \frac{-l_0}{2}, \frac{-l_1}{2}, \frac{l_2+1}{2}, \frac{l_3+1}{2}\right), \left(-l, \frac{l_0+1}{2}, \frac{l_1+1}{2}, \frac{-l_2}{2}, \frac{-l_3}{2}\right), \\ \left(-l, \frac{-l_0}{2}, \frac{l_1+1}{2}, \frac{-l_2}{2}, \frac{l_3+1}{2}\right), \left(-l, \frac{l_0+1}{2}, \frac{-l_1}{2}, \frac{l_2+1}{2}, \frac{-l_3}{2}\right), \\ \left(-l, \frac{-l_0}{2}, \frac{l_1+1}{2}, \frac{l_2+1}{2}, \frac{-l_3}{2}\right), \left(-l, \frac{l_0+1}{2}, \frac{-l_1}{2}, \frac{-l_2}{2}, \frac{l_3+1}{2}\right), \end{cases}$$

are permitted.

By a straightforward calculation, the dimension of direct sum of spaces W_d^{sym} of elliptic functions can be calculated. For the case $N = 1$, the dimension is

$$\begin{cases} 2k_0 + 1 & \tilde{l} \text{ is even and } k_0 + k_3 \geqslant \frac{\tilde{l}}{2} \\ \tilde{l} - 2k_3 + 1 & \tilde{l} \text{ is even and } k_0 + k_3 < \frac{\tilde{l}}{2} \\ 2k_0 + 1 & \tilde{l} \text{ is odd and } k_0 \geqslant \frac{\tilde{l}+1}{2} \\ \tilde{l} + 2 & \tilde{l} \text{ is odd and } k_0 < \frac{\tilde{l}+1}{2} \end{cases}$$

where $\tilde{l} = l_0 + l_1 + l_2 + l_3$, $k_0 = \max(l_0, l_1, l_2, l_3)$ and $k_3 = \min(l_0, l_1, l_2, l_3)$ (see also [17, 19]). For the case $N = 2$, the dimension is $(2l+1)^2 + \sum_{i=0}^{3} l_i(l_i+1)$ for all the cases $l, l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geqslant 0}$. Since the dimension is directly related to the genus of the spectral curve for the case $N = 1$ [19], the dimension for the case $N \geqslant 2$ might also play an important role.

4. Ruijsenaars models and Inozemtsev models

In [15], Ruijsenaars introduced a relativistic version of the Calogero–Moser–Sutherland model, which is called the Ruijsenaars model of type A_N these days. In [1], van Diejen introduced the BC_N Ruijsenaars-type model which has ten parameters $(\kappa; \mu, \nu_0, \bar{\nu}_0, \nu_1, \bar{\nu}_1, \nu_2, \bar{\nu}_2, \nu_3, \bar{\nu}_3)$, and Hikami and Komori constructed higher commuting operators using root algebra in [5, 7–9], that ensures the integrability.

The lowest operator of the BC_N (or $A_{2N}^{(2)}$) Ruijsenaars model is given as follows:

$$
\begin{aligned}
Y_1 = \sum_{j=1}^{N} &\left(\prod_{\substack{k=1\\k\neq j}}^{N} \frac{\theta_1(x_j - x_k - \mu)}{\theta_1(x_j - x_k)} \frac{\theta_1(x_j + x_k - \mu)}{\theta_1(x_j + x_k)} \right) \\
&\times \left(\prod_{r=0}^{3} \frac{\theta_{r+1}(x_j - \nu_r)}{\theta_{r+1}(x_j)} \frac{\theta_{r+1}(x_j + \kappa/2 - \bar{\nu}_r)}{\theta_{r+1}(x_j + \kappa/2)} \right) t_j(\kappa) \\
&+ \sum_{j=1}^{N} \left(\prod_{\substack{k=1\\k\neq j}}^{N} \frac{\theta_1(x_j + x_k + \mu)}{\theta_1(x_j + x_k)} \frac{\theta_1(x_j - x_k + \mu)}{\theta_1(x_j - x_k)} \right) \\
&\times \left(\prod_{r=0}^{3} \frac{\theta_{r+1}(x_j + \nu_r)}{\theta_{r+1}(x_j)} \frac{\theta_{r+1}(x_j - \kappa/2 + \bar{\nu}_r)}{\theta_{r+1}(x_j - \kappa/2)} \right) t_j(-\kappa) \\
&+ \sum_{p=0}^{3} \left(\frac{\pi}{\theta_1'(0)} \right)^2 \frac{2}{\theta_1(\mu)\theta_1(\kappa + \mu)} \left(\prod_{r=0}^{3} \theta_{r+1}\left(\kappa/2 + \nu_{\pi_p r}\right) \theta_{r+1}\left(\bar{\nu}_{\pi_p r}\right) \right) \\
&\times \left(\prod_{j=1}^{N} \frac{\theta_{p+1}(x_j - \kappa/2 - \mu)}{\theta_{p+1}(x_j - \kappa/2)} \frac{\theta_{p+1}(x_j + \kappa/2 + \mu)}{\theta_{p+1}(x_j + \kappa/2)} \right).
\end{aligned}
\tag{4.1}
$$

Here $\theta_j(x)$ $(j = 1, 2, 3, 4)$ is the Jacobi theta function (see (A.4)) and $t_i(\kappa)$ is a translation of the variable x_i by κ, i.e. $t_i(\kappa) f(x_1, \dots, x_i, \dots x_N) = f(x_1, \dots, x_i + \kappa, \dots x_N)$. π_r $(r = 0, 1, 2, 3)$ denotes the permutation $\pi_0 = \mathrm{id}$, $\pi_1 = (01)(23)$, $\pi_2 = (02)(13)$, $\pi_3 = (03)(12)$, where

$$
(ij)k = \begin{cases} k & k \neq i, j \\ j & k = i \\ i & k = j. \end{cases}
$$

Hikami and Komori showed that if $k = \left(2(N-1)\mu + \sum_{i=0}^{3}(\nu_i + \bar{\nu}_i)\right)\big/\kappa \in 2\mathbb{Z}_{\geqslant 0}$ then the operator Y_1 and higher commuting operators preserve the space of level k theta functions of type $A_{2N}^{(2)}$. They proved it using root algebra. Their presentation of the invariant subspace would be technical for non-experts. In this section, we describe them plainly.

The space of level k theta functions is defined as follows:

$$
Th_k^{W(B_N)} = \left\{ f : \mathbb{C}^N \to \mathbb{C} \;\middle|\; \begin{array}{l} \text{holomorphic, } W(B_N)\text{-invariant} \\ f(x+n) = f(x), \ (\forall n \in \mathbb{Z}^N) \\ f(x + n\tau) = f(x)\, \mathrm{e}^{-2\pi\sqrt{-1}k((x|n) + (n|n)\tau/2)} \end{array} \right\}
\tag{4.2}
$$

where $(x|y) = \sum_{i=1}^{N} x_i y_i$ for $x = (x_1, \dots, x_N) \in \mathbb{C}^N$ and $y = (y_1, \dots, y_N) \in \mathbb{C}^N$. A function $f(x_1, \dots, x_N)$ is $W(B_N)$-invariant if and only if the relations $f(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = f(x_1, \dots, x_N)$ for $\forall \sigma \in \mathfrak{S}_N$ and $f(x_1, \dots, x_i, \dots, x_N) = f(x_1, \dots, -x_i, \dots, x_N)$ for $\forall i \in \{1, \dots, N\}$ are satisfied. For the case $N = 1$, we obtain $\dim Th_{2l}^{W(B_1)} = l + 1$ for $l \in \mathbb{Z}_{\geqslant 0}$. Let $\theta^{(1)}(x), \dots, \theta^{(l+1)}(x)$ be a basis of $Th_{2l}^{W(B_1)}$. Then the space $Th_{2l}^{W(B_N)}$ is spanned by functions $\sum_{\sigma \in \mathfrak{S}_N} \theta^{(k_1)}(x_{\sigma(1)}) \dots \theta^{(k_N)}(x_{\sigma(N)})$ $(1 \leqslant k_1 \leqslant \cdots \leqslant k_N \leqslant l + 1)$. Therefore, $\dim Th_{2l}^{W(B_N)} = \frac{(l+N)!}{l!N!}$ for $l \in \mathbb{Z}_{\geqslant 0}$.

Proposition 4.1 (cf [8])**.** *If* $k = \left(2(N-1)\mu + \sum_{i=0}^{3}(\nu_i + \bar{\nu}_i)\right)\big/\kappa \in 2\mathbb{Z}_{\geqslant 0}$ *then the operator* Y_1 *preserves the space* $Th_k^{W(B_N)}$.

Proof. Let $f(x) \in Th_k^{W(B_N)}$. Then the function $Y_1 f(x)$ is $W(B_N)$-invariant. From the quasi-periodicity of $\theta_i(x)$ $(i = 0, 1, 2, 3)$ (see (A.5)) and $f(x)$ (see (4.2)), the function $Y_1 f(x)$ has a quasi-periodicity as condition (4.2) when $k = (2(N-1)\mu + \sum_{i=0}^{3}(\nu_i + \bar{\nu}_i))/\kappa$. Hence if we show the holomorphy of the function $Y_1 f(x)$ on $\mathbb{C}^N$, we have $Y_1 f(x) \in Th_k^{W(B_N)}$. Thus it is sufficient to show that the residues of the function $Y_1 f(x)$ at $x_j - x_k = 0$, $x_j + x_k = 0$ $(1 \leqslant j \neq k \leqslant N)$ and $x_j = 0, 1/2, (1+\tau)/2, \tau/2, \pm\kappa/2, 1/2 \pm \kappa/2, (1+\tau)/2 \pm \kappa/2, \tau/2 \pm \kappa/2$ $(1 \leqslant j \leqslant N)$ are zero. These are shown directly by using the quasi-periodicity of $f(x)$ (4.2) and $\theta_i(x)$ $(i = 0, 1, 2, 3)$ (A.5). Note that we rely on the condition $k \in 2\mathbb{Z}$ in this step. □

Let us consider the non-relativistic (difference-differential) limit of the Ruijsenaars model. It is known that the Inozemtsev model appears by this limit. Now we will exhibit it explicitly.

Let $a = -\mu/\kappa$, $b_0 = -(\nu_1 + \bar{\nu}_1)/2\kappa$, $b_1 = -(\nu_2 + \bar{\nu}_2)/2\kappa$, $b_2 = -(\nu_3 + \bar{\nu}_3)/2\kappa$, $b_3 = -(\nu_0 + \bar{\nu}_0)/2\kappa$ and

$$\Theta(x) = \prod_{1 \leqslant j<k \leqslant N} (\theta_1(x_j - x_k)\theta_1(x_j + x_k))^a \prod_{j=1}^{N} \theta_1(x_j)^{2b_0}\theta_2(x_j)^{2b_1}\theta_3(x_j)^{2b_2}\theta_0(x_j)^{2b_3}. \tag{4.3}$$

Assume $a \in \{-l, l+1\}$, $b_0 \in \{-l_0/2, (l_0+1)/2\}$, $b_1 \in \{-l_1/2, (l_1+1)/2\}$, $b_2 \in \{-l_2/2, (l_2+1)/2\}$ and $b_3 \in \{-l_3/2, (l_3+1)/2\}$.

As $\kappa \to 0$ while a, b_0, b_1, b_2, b_3 are fixed,

$$(-\Theta(x) \circ Y_1 \circ \Theta(x)^{-1} + C_0)/\kappa^2 \to H \tag{4.4}$$

where H is the Hamiltonian of the BC_N Inozemtsev model given in (2.1) and C_0 is a constant. Hence, we recover the Hamiltonian of the BC_N Inozemtsev model from an operator of the BC_N Ruijsenaars model via a limit $\kappa \to 0$.

Let us make a correspondence between the invariant spaces of theta functions on the BC_N Ruijsenaars model and the space related to the quasi-exact solvability on the BC_N Inozemtsev model.

Proposition 4.2. *Let $a = -\mu/\kappa$, $b_0 = -(\nu_1 + \bar{\nu}_1)/2\kappa$, $b_1 = -(\nu_2 + \bar{\nu}_2)/2\kappa$, $b_2 = -(\nu_3 + \bar{\nu}_3)/2\kappa$, $b_3 = -(\nu_0 + \bar{\nu}_0)/2\kappa$. Let $Th_{2k}^{W(B_N)}$ (4.2) be the theta-type invariant space of BC_N Ruijsenaars model, W_k^{sym} (3.11) be the invariant space of BC_N Inozemtsev model and $\Theta(x)$ be the function defined in (4.4). Assume $k = -((N-1)a + b_0 + b_1 + b_2 + b_3) \in \mathbb{Z}_{\geqslant 0}$.*

Then the map

$$\begin{array}{cccc} \phi: & Th_{2k}^{W(B_N)} & \to & W_k^{\mathrm{sym}} \\ & f(x_1, \ldots, x_N) & \mapsto & \Theta(x) f(x_1, \ldots, x_N) \end{array} \tag{4.5}$$

is an isomorphism of vector spaces.

Proof. Let us consider the correspondence between the space $Th_{2k}^{W(B_N)}$ and the space V_k^{sym}, where V_k^{sym} was defined in proposition 2.1.

Let $f(x_1, \ldots, x_N) \in Th_{2k}^{W(B_N)}$ and $g(x_1, \ldots, x_N) = f(x_1, \ldots, x_N)\Theta(x)\Phi(\wp(x_1), \ldots, \wp(x_N))^{-1}$, where the function $\Phi(z_1, \ldots, z_N)$ was defined in (2.2). From the condition $k \in \mathbb{Z}$, the function $g(x_1, \ldots, x_N)$ does not have branches on $\mathbb{C}^N$. It is seen that the function $g(x_1, \ldots, x_N)$ is doubly periodic, $W(B_N)$-invariant and may have poles only along $x_j = 0$ $(j = 1, \ldots, N)$ up to periods with degree at most k.

Hence there exists $\tilde{g}(z_1, \ldots, z_N) \in V_k^{\mathrm{sym}}$ such that $\tilde{g}(\wp(x_1), \ldots, \wp(x_N)) = g(x_1, \ldots, x_N)$ by a similar argument as in the proof of proposition 3.2.

By composing with the canonical map from V_k^{sym} to W_k^{sym}, we obtain $\phi\big(Th_{2k}^{W(B_N)}\big) \subset W_k^{\mathrm{sym}}$. It is obvious that the map ϕ is injective, and the dimension of $Th_{2k}^{W(B_N)}$ is equal to that of W_k^{sym}. Therefore, the map ϕ is bijective. □

In proposition 4.2, we have established that the theta-type invariant space $Th_{2k}^{W(B_N)}$ of the BC_N Ruijsenaars model corresponds to the space W_k^{sym} which is related to the quasi-exact solvability of the BC_N Inozemtsev model.

5. Degenerate Inozemtsev model

5.1. Trigonometric BC_N Inozemtsev model

In [16], Sasaki and Takasaki considered degenerate BC_N Inozemtsev models and their quasi-exact solvability. They are also considered for the case of type A_N.

In this section, we consider the degeneration of the BC_N Inozemtsev model and show that the finite-dimensional invariant spaces for the elliptic BC_N Inozemtsev model tend to the spaces introduced by Sasaki and Takasaki by the degeneration.

The Hamiltonian of the trigonometric (or degenerate) BC_N Inozemtsev model is given as follows:

$$H^{(\mathrm{D})} = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2l(l+1)\sum_{1\leqslant j<k\leqslant N}\left(\frac{\pi^2}{\sin^2\pi(x_j-x_k)}+\frac{\pi^2}{\sin^2\pi(x_j+x_k)}\right) + \sum_{j=1}^{N}\left(\frac{\pi^2 l_0(l_0+1)}{\sin^2\pi x_j}+\frac{\pi^2 l_1(l_1+1)}{\cos^2\pi x_j}+\tilde{c}_1\cos 2\pi x_j+\tilde{c}_2\cos 4\pi x_j\right) \tag{5.1}$$

where $l, l_0, l_1, \tilde{c}_1, \tilde{c}_2$ are coupling constants.

This model is known to be quantum integrable. In [13], Oshima gave the explicit expression of commuting operators of conserved quantities.

Set

$$\Phi_D(x) = \left|\exp\left(-\frac{\tilde{a}}{2}\sum_{j=1}^{N}\cos 2\pi x_j\right)\prod_{j=1}^{N}(\sin\pi x_j)^{l_0+1}(\cos\pi x_j)^{l_1+1} \times \prod_{1\leqslant j_1<j_2\leqslant N}\left(\sin\pi\left(x_{j_1}-x_{j_2}\right)\sin\pi(x_{j_1}+x_{j_2})\right)^{l+1}\right|. \tag{5.2}$$

Let $W_L^{(\mathrm{D})}$ be the vector space spanned by functions $\Phi_D(x)(\sin\pi x_1)^{2m_1}(\sin\pi x_2)^{2m_2}\cdots(\sin\pi x_N)^{2m_N}$ such that $m_i\in\{0,1,\ldots,L\}$ for all i, and $W_L^{(\mathrm{D}),\mathrm{sym}}$ be the set of $\mathfrak{S}_N$-invariant elements in $W_L^{(\mathrm{D})}$.

The following proposition is essentially shown in [16].

Proposition 5.1 (cf [16])**.** *The Hamiltonian $H^{(\mathrm{D})}$ (5.1) preserves the space $W_L^{(\mathrm{D}),\mathrm{sym}}$, if $L\in\mathbb{Z}_{\geqslant 0}$, $\tilde{c}_2=-\frac{\pi^2\tilde{a}^2}{2}$ and $\tilde{c}_1=2\tilde{a}\pi^2(2L+l_0+l_1+3+2(N-1)(l+1))$.*

Proof. We set $W_0=\log\Phi_D(x)$. Then we have

$$\sum_{j=1}^{N}\left(\left(\frac{\partial W_0}{\partial x_j}\right)^2+\frac{\partial^2 W_0}{\partial x_j^2}\right) = \sum_{1\leqslant j<k\leqslant N}\left(\frac{2\pi^2 l(l+1)}{\sin^2\pi(x_j-x_k)}+\frac{2\pi^2 l(l+1)}{\sin^2\pi(x_j+x_k)}\right) + \sum_{j=1}^{N}\left(\frac{\pi^2 l_0(l_0+1)}{\sin^2\pi x_j}+\frac{\pi^2 l_1(l_1+1)}{\cos^2\pi x_j}+\tilde{c}_3\cos 2\pi x_j-\frac{\pi^2\tilde{a}^2}{2}\cos 4\pi x_j\right)+C_0 \tag{5.3}$$

where C_0 is a constant term and $\tilde{c}_3 = 2\tilde{a}\pi^2(l_0 + l_1 + 3 + 2(N-1)(l+1))$. By comparing with the Hamiltonian in [16, (7.1)] and its corresponding 'exactly solvable sector' [16, (7.13)], we obtain the proposition. □

5.2. Degeneration

In this section, we consider the degeneration (the trigonometric limit) $\tau \to \sqrt{-1}\infty$ and see the correspondences of Hamiltonians and their invariant spaces between the nondegenerate model and the degenerate one.

Let l, l_0, l_1, l_2, l_3 be the coupling constants of the elliptic Inozemtsev model (see (2.1)). We adopt the following limits of coupling constants as $\tau \to \sqrt{-1}\infty$:

- l, l_0, l_1: fixed;
- $l_2 = \frac{\tilde{a}}{8}p^{-1} + \tilde{b}$ and $l_3 = -\frac{\tilde{a}}{8}p^{-1} + \tilde{b}$, where $p = \exp(\pi\sqrt{-1}\tau)$. Here we note that $p \to 0$ as $\tau \to \sqrt{-1}\infty$. Then the Hamiltonian H of the elliptic Inozemtsev model (see (2.1)) tends to the Hamiltonian $H^{(\mathrm{D})}$ of the trigonometric Inozemtsev model (see (5.1)). More precisely,

$$H + \frac{\pi^2}{3}(l_2(l_2+1) + l_3(l_3+1)) \to -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{1\leqslant j<k\leqslant N}\left(\frac{2\pi^2 l(l+1)}{\sin^2\pi(x_j - x_k)} + \frac{2\pi^2 l(l+1)}{\sin^2\pi(x_j + x_k)}\right) + \sum_{j=1}^{N}\left(\frac{\pi^2 l_0(l_0+1)}{\sin^2\pi x_j} + \frac{\pi^2 l_1(l_1+1)}{\cos^2\pi x_j} + 2\pi^2\tilde{a}(2\tilde{b}+1)\cos 2\pi x_j - \frac{\pi^2\tilde{a}^2}{2}\cos 4\pi x_j\right) + C_1 \tag{5.4}$$

as $p \to 0$, where C_1 is a constant.

Let us observe how the invariant space varies as $p \to 0$. Since $\tilde{b} = \frac{l_2+l_3}{2}$, the Hamiltonian H (see (2.1)) preserves the space W_L^{sym} (see (3.11)) for $a = l+1, b_1 = \frac{l_1+1}{2}, b_2 = \frac{-l_2}{2}$ and $b_3 = \frac{-l_3}{2}$ if $L = -(N-1)(l+1) - \frac{l_0+l_1+2}{2} + \tilde{b} \in \mathbb{Z}_{\geqslant 0}$.

We consider the limit $p \to 0$. Note that if $L = -(N-1)(l+1) - \frac{l_0+l_1+2}{2} + \tilde{b} \in \mathbb{Z}_{\geqslant 0}$ then the Hamiltonian H preserves the space W_L^{sym} whenever p varies.

Let $\Phi(z) = \prod_{1\leqslant j<k\leqslant N}(z_j - z_k)^{l+1}\prod_{j=1}^{N}(z_j - e_1)^{\frac{l_1+1}{2}}(z_j - e_2)^{\frac{-l_2}{2}}(z_j - e_3)^{\frac{-l_3}{2}}$ be the function defined in (2.2) for $a = l+1, b_1 = \frac{l_1+1}{2}, b_2 = \frac{-l_2}{2}, b_3 = \frac{-l_3}{2}$. Then $\Phi(\wp(x_1), \ldots, \wp(x_N)) \to C_3\Psi_D(x)$ as $\tau \to \sqrt{-1}\infty$, where C_3 is a constant and

$$\Psi_D(x) = \prod_{j=1}^{N}(\sin\pi x_j)^{-2(N-1)(l+1)-(l_1+1)+2\tilde{b}}(\cos\pi x_j)^{l_1+1} \times \prod_{1\leqslant j_1<j_2\leqslant N}\left(\sin\pi\left(x_{j_1} - x_{j_2}\right)\sin\pi\left(x_{j_1} + x_{j_2}\right)\right)^{l+1}\exp\left(-\frac{\tilde{a}}{2}\sum_{j=1}^{N}\cos 2\pi x_j\right). \tag{5.5}$$

Set $t(x) = \frac{\pi^2}{\sin^2\pi x} - \frac{\pi^2}{3}$. Let $\tilde{W}_L^{(\mathrm{D})}$ be the vector space spanned by functions $\Psi_D(x)t(x_1)^{m_1}t(x_2)^{m_2}\cdots t(x_N)^{m_N}$ such that $m_i \in \{0, 1, \ldots, L\}$ for all i, and $\tilde{W}_L^{(\mathrm{D}),\mathrm{sym}}$ be the set of $\mathfrak{S}_N$-invariant elements in $\tilde{W}_L^{(\mathrm{D})}$.

As $p \to 0$, the vector space W_L^{sym} tends to the space $\tilde{W}_L^{(\mathrm{D}),\mathrm{sym}}$, and the operator which appears on the right-hand side of (5.4) preserves the space $\tilde{W}_L^{(\mathrm{D}),\mathrm{sym}}$ if $L = -(N-1)(l+1)$

$-\frac{l_0+l_1+2}{2}+\tilde{b}\in\mathbb{Z}_{\geqslant 0}$. If $l_0+1=2\tilde{b}-2(N-1)(l+1)-(l_1+1)$ then it is seen that $\tilde{W}_L^{(\mathrm{D}),\mathrm{sym}}=W_L^{(\mathrm{D}),\mathrm{sym}}$. Therefore, we recover proposition 5.1 by the trigonometric limit.

In summary, by the trigonometric limit we have shown that some finite-dimensional invariant spaces of the Hamiltonian H of the elliptic model (see (2.1)) tend to the invariant space of Sasaki and Takasaki which is related to the quasi-exact solvability.

In [13], Oshima described the limit procedure of the commuting operators of conserved quantities. By applying Oshima's result and proposition 3.4 in this paper, it follows that the commuting operators of the trigonometric BC_N Inozemtsev model also preserve the space $W_L^{(\mathrm{D}),\mathrm{sym}}$.

Hence, we established that the commuting operators of the trigonometric BC_N Inozemtsev model also preserve the space related to the quasi-exact solvability.

6. Concluding remarks

In this paper, quasi-exact solvability for the BC_N Inozemtsev model is proved not only for the Hamiltonian but also for commuting operators of conserved quantities. It is seen that the theta-type invariant spaces for the BC_N Ruijsenaars model correspond to the spaces which are related to the quasi-exact solvability for the BC_N Inozemtsev model, and the degeneration of the BC_N Inozemtsev model (especially for its quasi-exact solvability) is clarified.

In papers [2, 3, 20], several models which are related to the Inozemtsev model are studied. It would be interesting to link their results with ours. In [10, 18], the method of perturbation for the elliptic Calogero–Moser–Sutherland models from the trigonometric models is introduced. For the Hamiltonian of the BC_N Inozemtsev model, holomorphy of perturbation from the trigonometric model can be established. The relationship between the perturbation and the complete integrability should be clarified. More precisely, holomorphy of perturbation for commuting operators of conserved quantities should be shown, although it is not successful as of now.

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Appendix

This appendix presents the definitions of and formulae for elliptic functions.

Let ω_1 and ω_3 be complex numbers such that the value ω_3/ω_1 is an element of the upper half-plane. The Weierstrass $\wp$-function is defined as follows:

$$\wp(x)=\wp(x|2\omega_1,2\omega_3)\\ =\frac{1}{x^2}+\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}}\left(\frac{1}{(x-2m\omega_1-2n\omega_3)^2}-\frac{1}{(2m\omega_1+2n\omega_3)^2}\right). \tag{A.1}$$

Setting $\omega_2=-\omega_1-\omega_3$ and

$$e_i=\wp(\omega_i)\qquad(i=1,2,3). \tag{A.2}$$

yields the relations

$$
\begin{aligned}
& e_1+e_2+e_3=0 \qquad \wp(x+2\omega_j)=\wp(x) \qquad (j=1,2,3) \\
& \frac{\wp''(x)}{\wp'(x)^2}=\frac{1}{2}\left(\frac{1}{\wp(x)-e_1}+\frac{1}{\wp(x)-e_2}+\frac{1}{\wp(x)-e_3}\right) \\
& \wp(x+y)=\frac{1}{4}\left(\frac{\wp'(x)+\wp'(y)}{\wp(x)-\wp(y)}\right)^2-\wp(x)-\wp(y) \\
& \wp(x+y)+\wp(x-y)=\frac{\wp'(x)^2+\wp'(y)^2}{2(\wp(x)-\wp(y))^2}-2\wp(x)-2\wp(y) \\
& \wp(x+\omega_i)=e_i+\frac{(e_i-e_{i'})(e_i-e_{i''})}{\wp(x)-e_i} \qquad (i=1,2,3)
\end{aligned}
\tag{A.3}
$$

where $i', i'' \in \{1, 2, 3\}$ with $i' < i''$, $i \neq i'$ and $i \neq i''$.

Let $\omega_1 = 1/2$ and $\tau = \omega_3/\omega_1$. The Jacobi theta functions are defined by

$$
\begin{aligned}
& \theta_1(x)=2\sum_{n=1}^{\infty}(-1)^{n-1}\,\mathrm{e}^{\tau\pi\sqrt{-1}\left(n-\frac{1}{2}\right)^2}\sin(2n-1)\pi x \\
& \theta_2(x)=2\sum_{n=1}^{\infty}\mathrm{e}^{\tau\pi\sqrt{-1}\left(n-\frac{1}{2}\right)^2}\cos(2n-1)\pi x \\
& \theta_3(x)=1+2\sum_{n=1}^{\infty}\mathrm{e}^{\tau\pi\sqrt{-1}n^2}\cos 2n\pi x \\
& \theta_0(x)=\theta_4(x)=1+2\sum_{n=1}^{\infty}(-1)^n\,\mathrm{e}^{\tau\pi\sqrt{-1}n^2}\cos 2n\pi x.
\end{aligned}
\tag{A.4}
$$

Then the following relations are satisfied:

$$
\begin{aligned}
& \wp(x)=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\theta_1(x)+\mathrm{const} \\
& \wp(x+\omega_i)=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\theta_{i+1}(x)+\mathrm{const} \qquad (i=1,2,3) \\
& \theta_1(x)=-\theta_1(-x) \qquad \theta_i(x)=\theta_i(-x) \qquad (i=0,2,3) \\
& \theta_i(x+1)=-\theta_i(x) \qquad (i=1,2) \qquad \theta_i(x+1)=\theta_i(x) \qquad (i=0,3) \\
& \theta_i(x+\tau)=-\mathrm{e}^{-\pi\sqrt{-1}(2x+\tau)}\theta_i(x) \qquad (i=0,1) \\
& \theta_i(x+\tau)=\mathrm{e}^{-\pi\sqrt{-1}(2x+\tau)}\theta_i(x) \qquad (i=2,3) \\
& \theta_1(2x)\theta_2(0)\theta_3(0)\theta_0(0)=2\theta_1(x)\theta_2(x)\theta_3(x)\theta_0(x) \\
& \theta_1'(0)=\pi\theta_2(0)\theta_3(0)\theta_0(0).
\end{aligned}
\tag{A.5}
$$

Let $p = \exp(\pi\sqrt{-1}\tau)$. The expansions of the functions $\wp(x)$, $\wp\left(x+\frac{1}{2}\right)$, $\wp\left(x+\frac{\tau}{2}\right)$ and $\wp\left(x+\frac{1+\tau}{2}\right)$ in p are given as follows:

$$
\begin{aligned}
& \wp(x)=\frac{\pi^2}{\sin^2(\pi x)}-\frac{\pi^2}{3}-8\pi^2\sum_{n=1}^{\infty}\frac{np^{2n}}{1-p^{2n}}(\cos 2n\pi x-1) \\
& \wp\left(x+\frac{1}{2}\right)=\frac{\pi^2}{\cos^2(\pi x)}-\frac{\pi^2}{3}-8\pi^2\sum_{n=1}^{\infty}\frac{np^{2n}}{1-p^{2n}}((-1)^n\cos 2n\pi x-1)
\end{aligned}
$$

$$\wp\left(x+\frac{\tau}{2}\right) = -\frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty} np^n \frac{\cos 2\pi nx - p^n}{1-p^{2n}}$$
$$\wp\left(x+\frac{1+\tau}{2}\right) = -\frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty} np^n \frac{(-1)^n\cos 2\pi nx - p^n}{1-p^{2n}}. \tag{A.6}$$

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